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The structure of spectral problems and geometry: hyperbolic surfaces in $\mathbf{E}^3$

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Abstract
Working in the framework of Sym's soliton surfaces approach we point out that some simple assumptions about the structure of linear (spectral) problems of the theory of solitons lead uniquely to the geometry of some special immersions. In this paper we consider general $su(2)$ spectral problems. Under some very weak assumptions they turn out to be associated with hyperbolic surfaces (surfaces of negative Gaussian curvature) immersed in three-dimensional Euclidean space, and especially with the so-called Bianchi surfaces.

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1. Introduction

It is very well known that some problems in the geometry of immersions are in a close relationship with the theory of completely integrable systems. Indeed, the famous Bäcklund transformation which is a fundamental property of integrable systems first appeared in a geometrical context. Bäcklund constructed a transformation between pseudospherical surfaces in three-dimensional Euclidean space $\mathbf{E}^3$. In the past few years a lot of papers on numerous connections of integrability and the differential geometry appeared (see, for instance, [1–7]). It is intriguing that many problems of classical differential geometry (which often can be traced back to the 19th century [8, 9]) can be interpreted in the language of the modern theory of solitons.

We use the soliton surfaces approach, i.e. given a matrix wavefunction Ψ (the fundamental solution of a spectral problem) we define an immersion using the so-called Sym (or Sym–Tafel) formula [10, 11]:

$$F := \Psi^{-1}\Psi_{,\lambda} \tag{1}$$

where λ is the spectral parameter. F defined in this way is, obviously, a matrix. However, treating the matrix as an element of some linear space (a subspace of $gl(n, \mathbf{C})$) we can identify F with an immersion into this linear space (more precisely, we have a λ-family of such immersions). If the linear space has the structure of some Lie algebra, then we can use a natural scalar product—the Killing–Cartan form.

In a series of papers it has been shown that the soliton surfaces approach leads to a surprisingly rich family of immersions which are interesting both for geometrical and physical reasons [1, 3, 11–13]. The Sym formula enables one to apply numerous techniques of the theory of solitons where the wavefunction appears explicitly (inverse scattering transform, loop group approach, Riemann theta function solutions, Darboux–Bäcklund transformation, etc) in the geometric context.

In this paper we confine ourselves to the Lie group $SU(2)$, namely $\Psi = \Psi(x, y; \lambda) \in SU(2)$ and $F = F(x, y; \lambda) \in su(2)$. We recall that the Lie algebra $su(2)$ can be identified with the Euclidean space $\mathbf{E}^3$ (compare remark 2).

We are going to show that the geometric properties of the immersion F can be related in a natural way to quite general algebraic structure of the associated spectral problem (the most important part of this structure is the corresponding loop group). Special attention is given to the fact that some very simple assumptions on the structure of the associated linear problem lead uniquely to the specific class of immersions. The important motivation behind our research consists in preparing a ground for future discretization of the considered spectral problems.

From geometrical point of view, we can say that the known results usually concern some specific system of coordinates, some specific gauge, etc. In our paper we present a novel observation that the structure of a spectral problem (appropriately defined) can uniquely yield some classes of immersions in arbitrary coordinates.

It would be interesting to connect our results with the approach which is from the beginning coordinate independent, namely the so-called CC-ideals method. In this approach the considered system of nonlinear partial differential equations is written as a set of differential forms (*constant coefficients ideal*) [14–17]. Then a convenient parametrization of the manifold defined by the considered CC-ideal is chosen. It seems that discretization procedures have not yet been studied within this approach.

The case considered in the present paper (the $su(2)$ loop algebra and, especially, the twisted $su(2)$ loop algebra) is associated with hyperbolic surfaces in $\mathbf{E}^3$ (i.e. surfaces of negative Gaussian curvature at each point). Our main result consists in showing that very weak assumptions (twisted $su(2)$ non-isospectral linear problem with two poles, at $\lambda = 0$ and $\lambda = \infty$) uniquely lead to the so-called Bianchi surfaces, i.e. surfaces with the Gaussian curvature of the form $K = -1/\rho^2$ where $\rho_{,xy} = 0$ (x, y are asymptotic coordinates). We do not need any assumptions about the dependence of λ on x, y. The most general dependence which is consistent with the compatibility conditions leads exactly to the class of Bianchi surfaces. The isospectral case corresponds to pseudospherical surfaces (i.e. $K = \text{const} < 0$) in arbitrary coordinates.

Bianchi surfaces and the associated nonlinear system have been recently intensively studied in the framework of the soliton theory [18–22]. At the end of this paper some physical applications are discussed. Special attention is given to the relations between the so-called pumped Maxwell–Bloch system and Bianchi surfaces.

2. Pseudospherical surfaces, asymptotic coordinates

Consider the linear problem

$$
\begin{aligned}
\Psi_{,x} &= U\Psi \qquad & U &:= A\mathbf{e}_1 + \lambda(B\mathbf{e}_2 + C\mathbf{e}_3) \\
\Psi_{,y} &= V\Psi \qquad & V &:= P\mathbf{e}_1 + \frac{1}{\lambda}(Q\mathbf{e}_2 + R\mathbf{e}_3)
\end{aligned}
\tag{2}
$$

where A, B, C, P, Q, R are some scalar functions of x, y and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an orthonormal basis in $su(2)$ (one can take, for instance, $\mathbf{e}_k := -\frac{\mathrm{i}}{2}\sigma_k$, where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices). In particular, $[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$ etc. The function Ψ is the fundamental solution (matrix-valued) and we can confine ourselves to $\Psi(x, y; \lambda) \in SU(2)$. The only restriction on the scalar coefficients is the existence of a non-zero solution Ψ, i.e. the compatibility conditions.

A remark concerning the terminology is in order. We will refer to linear systems such as (2) as 'spectral problems'. In the theory of solitons it is more popular to use this name for only one of the linear equations. All of them are known rather as the 'linear problem' or (in the case of two equations) the 'Lax pair'. However, for many linear problems (including (2)) all equations are evidently on equal footing and there are no necessary reasons to distinguish one of them as the spectral problem.

The following analytic and algebraic properties are the most important ingredients of the structure of the spectral problem (2). First, U and V are rational functions of λ, U has a simple pole at $\lambda = \infty$, while V has a simple pole at $\lambda = 0$. Second, U and V can be considered as maps from $\mathbf{R}^2$ (or a region of $\mathbf{R}^2$) into the *twisted* loop algebra $su(2)$ which is defined by the following constraints:

$$U(-\lambda)\mathbf{e}_1 = \mathbf{e}_1 U(\lambda) \qquad V(-\lambda)\mathbf{e}_1 = \mathbf{e}_1 V(\lambda). \tag{3}$$

One can easily see that these properties imply uniquely the form (2) of the spectral problem.

Proposition 1. *Let $F = \Psi^{-1}\Psi_{,\lambda}$ where Ψ is an $SU(2)$-solution to (2). Then $F = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$ where f_k are real functions of x, y, λ and (f_1, f_2, f_3) is a pseudospherical immersion in $\mathbf{E}^3$ with Gaussian curvature $K = -1/\lambda^2$. What is more, x, y are asymptotic coordinates on the pseudospherical surface.*

Proof. We compute tangent vectors

$$F_{,x} = \Psi^{-1}U_{,\lambda}\Psi = \Psi^{-1}(B\mathbf{e}_2 + C\mathbf{e}_3)\Psi$$
$$F_{,y} = \Psi^{-1}V_{,\lambda}\Psi = -\lambda^{-2}\Psi^{-1}(Q\mathbf{e}_2 + R\mathbf{e}_3)\Psi.$$

The normalized skew product of tangent vectors yields the normal vector $N = \Psi^{-1}\mathbf{e}_1\Psi$. Then

$$N_{,x} = \Psi^{-1}[\mathbf{e}_1, U]\Psi = \Psi^{-1}(\lambda B\mathbf{e}_3 - \lambda C\mathbf{e}_2)\Psi$$
$$N_{,y} = \Psi^{-1}[\mathbf{e}_1, V]\Psi = \Psi^{-1}(\lambda^{-1}Q\mathbf{e}_3 - \lambda^{-1}R\mathbf{e}_2)\Psi.$$

Thus we are in a position to calculate explicitly fundamental forms, i.e. $I = \langle \mathrm{d}F|\mathrm{d}F\rangle$ and $II = -\langle \mathrm{d}F|\mathrm{d}N\rangle$. Namely

$$I = (B^2 + C^2)\,\mathrm{d}x^2 - \frac{2}{\lambda^2}(BQ + CR)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\lambda^4}(R^2 + Q^2)\,\mathrm{d}y^2$$
$$II = \frac{2}{\lambda}(BR - CQ)\,\mathrm{d}x\,\mathrm{d}y$$

and the Gaussian curvature of F (given by $K = \det(I)/\det(II)$) is easily found to be $K = -\lambda^{-2}$ which ends the proof. □

Solving part of the compatibility conditions we will parametrize the Lax pair (2) in a more convenient way. The compatibility conditions for (2) read

$$\begin{aligned} &A_{,y} - P_{,x} + BR - CQ = 0 \\ &(\lambda B)_{,y} + \lambda CP - \lambda^{-1}AR - (\lambda^{-1}Q)_{,x} = 0 \\ &(\lambda C)_{,y} - \lambda BP + \lambda^{-1}AQ - (\lambda^{-1}R)_{,x} = 0. \end{aligned} \tag{4}$$

Because $\lambda = \text{const}$ we can rewrite the last two equations as follows:

$$Q_{,x} = -AR \qquad R_{,x} = AQ \qquad B_{,y} = -CP \qquad C_{,y} = BP. \tag{5}$$

It is not difficult to derive from (5) that

$$(Q^2 + R^2)_{,x} = 0 \qquad (B^2 + C^2)_{,y} = 0.$$

Hence the system (5) can be solved to give

$$\begin{array}{lll} A = \beta_{,x} & B = f(x)\cos\alpha & C = f(x)\sin\alpha \\ P = \alpha_{,y} & Q = g(y)\cos\beta & R = g(y)\sin\beta \end{array}$$

where f, g are functions of one variable and α, β are functions satisfying the first equation of the system (4), i.e.

$$(\alpha - \beta)_{,xy} + fg\sin(\alpha - \beta) = 0. \tag{6}$$

Denoting $\varphi := \alpha - \beta$ and reparametrizing asymptotic lines $\mathrm{d}u := f(x)\,\mathrm{d}x$, $\mathrm{d}v := -g(y)\,\mathrm{d}y$, we obtain

$$I = \mathrm{d}u^2 + \frac{2}{\lambda^2}\cos\varphi\,\mathrm{d}u\,\mathrm{d}v + \frac{1}{\lambda^4}\,\mathrm{d}v^2 \qquad II = \frac{2}{\lambda}\sin\varphi\,\mathrm{d}u\,\mathrm{d}v. \tag{7}$$

The function $\varphi = \varphi(u, v)$ satisfies the sine-Gordon equation

$$\varphi_{,uv} = \sin\varphi. \tag{8}$$

Note that from the physical point of view asymptotic coordinates are (in this case) light cone coordinates.

Remark 1. If the spectral problem (2) is of a more restricted form, namely

$$A = \varphi_{,x} \qquad B = 0 \qquad C = 1 \qquad P = 0 \qquad Q = \sin\varphi \qquad R = -\cos\varphi$$

then we can recognize in (2) the standard Lax pair for the sine-Gordon equation (8). The observation that the formula (1) yields in this case pseudospherical surfaces is due to Sym [23].

Remark 2. In what follows the $su(2)$-valued $F = F(x, y; \lambda)$ given by $F = \Psi^{-1}\Psi_{,\lambda}$ will always be treated as a λ-family of surfaces in $\mathbf{E}^3$ according to the decomposition $F = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$, i.e.

$$su(2) \ni F \longleftrightarrow (f_1, f_2, f_3) \in \mathbf{E}^3.$$

The basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ corresponds to a right-oriented orthonormal basis in $\mathbf{E}^3$. The commutator of $su(2)$-matrices is identified with the vector (skew) product in $\mathbf{E}^3$.

Remark 3. Here and in the following we use the notation $F(\lambda) = \Psi^{-1}\Psi_{,\lambda}$ instead of the more precise but less convenient notation $F(\lambda_0) = \Psi^{-1}\Psi_{,\lambda}|_{\lambda=\lambda_0}$.

3. Pseudospherical surfaces, arbitrary coordinates

Consider the most general linear problem $\Psi_{,k} = U_k\Psi$ ($k = 1, 2$) subject only to the following conditions:

- U_k are rational in λ with simple poles at $\lambda = 0$ and $\lambda = \infty$,
- U_k are real linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (generators of $su(2)$),
- $U_k(-\lambda) = \mathbf{e}_1 U_k(\lambda)\mathbf{e}_1^{-1}$.

In other words, U_1 and U_2 are functions with values in the twisted $su(2)$ loop algebra with at most two simple poles, at $\lambda = 0$ and $\lambda = \infty$. Such a linear problem has the following explicit representation:

$$\Psi_{,k} = \left(a_k \mathbf{e}_1 + \left(b_k\lambda + \frac{b'_k}{\lambda} \right) \mathbf{e}_2 + \left(c_k\lambda + \frac{c'_k}{\lambda} \right) \mathbf{e}_3 \right) \Psi \tag{9}$$

where $a_k, b_k, b'_k, c_k, c'_k$ are real functions of x^1, x^2. Defining F by the formula (1) we compute

$$F_{,k} = \Psi^{-1} \left(\left(b_k - \frac{b'_k}{\lambda^2} \right) \mathbf{e}_2 + \left(c_k - \frac{c'_k}{\lambda^2} \right) \mathbf{e}_3 \right) \Psi. \tag{10}$$

Therefore, the normal vector is given by $N = \Psi^{-1}\mathbf{e}_1\Psi$ and

$$N_{,k} = \Psi^{-1}[\mathbf{e}_1, U_k]\Psi = \Psi^{-1} \left(- \left(c_k\lambda + \frac{c'_k}{\lambda} \right) \mathbf{e}_2 + \left(b_k\lambda + \frac{b'_k}{\lambda} \right) \mathbf{e}_3 \right) \Psi. \tag{11}$$

The fundamental forms, $I = \langle \mathrm{d}F | \mathrm{d}F \rangle$ and $II = -\langle \mathrm{d}F | \mathrm{d}N \rangle$, can be computed in the straightforward way:

$$I = \left(B_1^2 + C_1^2 \right) \mathrm{d}x^2 + 2(B_1 B_2 + C_1 C_2)\, \mathrm{d}x\, \mathrm{d}y + \left(B_2^2 + C_2^2 \right) \mathrm{d}y^2 \tag{12}$$

where $x := x^1$, $y := x^2$, $B_k := b_k - b'_k\lambda^{-2}$, $C_k := c_k - c'_k\lambda^{-2}$, and

$$II = \frac{2}{\lambda}((b_1\, \mathrm{d}x + b_2\, \mathrm{d}y)(c'_1\, \mathrm{d}x + c'_2\, \mathrm{d}y) - (b'_1\, \mathrm{d}x + b'_2\, \mathrm{d}y)(c_1\, \mathrm{d}x + c_2\, \mathrm{d}y)). \tag{13}$$

In particular,

$$\begin{aligned} \det(I) &= (B_1 C_2 - B_2 C_1)^2 \\ \det(II) &= \frac{1}{\lambda^2}(4(b_1 c'_1 - b'_1 c_1)(b_2 c'_2 - b'_2 c_2) - (b_2 c'_1 + b_1 c'_2 - b'_2 c_1 - b'_1 c_2)^2). \end{aligned} \tag{14}$$

To obtain further simplification we have to use compatibility conditions for the system (9):

$$\begin{aligned} &\lambda^2 \mathbf{e}_1 (b_2 c_1 - b_1 c_2) = 0 \\ &\lambda^{-2} \mathbf{e}_1 (b'_2 c'_1 - b'_1 c'_2) = 0 \\ &\mathbf{e}_1 (a_{1,y} - a_{2,x} + c_1 b'_2 + c'_1 b_2 - b_1 c'_2 - b'_1 c_2) = 0 \\ &\lambda \mathbf{e}_2 (b_{1,y} - b_{2,x} + a_1 c_2 - c_1 a_2) = 0 \\ &\lambda^{-1} \mathbf{e}_2 (b'_{1,y} - b'_{2,x} + a_1 c'_2 - c'_1 a_2) = 0 \\ &\lambda \mathbf{e}_3 (c_{1,y} - c_{2,x} + b_1 a_2 - b_2 a_1) = 0 \\ &\lambda^{-1} \mathbf{e}_3 (c'_{1,y} - c'_{2,x} + b'_1 a_2 - b'_2 a_1) = 0. \end{aligned} \tag{15}$$

In fact, only the first two equations (for coefficients by highest and lowest powers of λ) are useful. They imply

$$\begin{array}{llll} b_1 = \nu_1 b & c_1 = \nu_1 c & b'_1 = \nu'_1 b' & c'_1 = \nu'_1 c' \\ b_2 = \nu_2 b & c_2 = \nu_2 c & b'_2 = \nu'_2 b' & c'_2 = \nu'_2 c' \end{array} \tag{16}$$

where $\nu_1, \nu_2, \nu'_1, \nu'_2, b, c, b', c'$ are some functions. Now the straightforward computation yields

$$\begin{aligned} \det(I) &= \frac{1}{\lambda^4}(\nu_1\nu'_2 - \nu_2\nu'_1)^2 (bc' - cb')^2 \\ \det(II) &= -\frac{1}{\lambda^2}(\nu_1\nu'_2 - \nu_2\nu'_1)^2 (bc' - cb')^2 \end{aligned} \tag{17}$$

which means that the Gaussian curvature $K = \det(I)/\det(II) = -\lambda^{-2}$ is constant and negative. Therefore we have proved the following proposition.

Proposition 2. *Let $\Psi_{,k} = U_k\Psi$ $(k = 1, 2)$ where U_1, U_2 assume values in the twisted $su(2)$ loop algebra and have at most two simple poles, at $\lambda = 0$ and $\lambda = \infty$. Then, $F = \Psi^{-1}\Psi_{,\lambda}$ has the constant Gaussian curvature $K = -1/\lambda^2$.*

In the smooth case this result can be treated just as a curiosity. Our main motivation to study possibly general spectral problems is to obtain more general procedures of discretization for surfaces.

As a rule, the discrete geometry is closely related to a choice of very special coordinates (e.g., pseudospherical surfaces were discretized in asymptotic coordinates, compare [24] and references cited therein).

The results of the present paper give some hopes to extend the notions of discrete geometry on arbitrary coordinates. Namely, if we are able to find a discretization of the given linear problem preserving all its analytical and loop group properties, then we can expect that also the basic geometrical properties (concerning the immersions produced by the Sym–Tafel formula) are preserved. Such a procedure turned out to be successful in the case of pseudospherical surfaces in asymptotic coordinates [24] and isothermic surfaces in curvature coordinates [25]. In both cases the correctness of this discretization has been confirmed by other (e.g., geometrical) arguments.

The discretization of the linear problem (9) preserving its analytical structure and loop group properties (i.e. preserving the conditions listed at the beginning of section 3) reads

$$T_k\Psi = U_k\Psi \qquad U_k := \mathrm{i}a_k\sigma_1 + \mathrm{i}\left(b_k\lambda + \frac{b_k'}{\lambda}\right)\sigma_2 + \mathrm{i}\left(c_k\lambda + \frac{c_k'}{\lambda}\right)\sigma_3 + d_k \tag{18}$$

where $k = 1, 2$ and $\Psi, a_k, b_k, b_k', c_k, c_k', d_k$ are functions of discrete variables m, n and of (continuous) parameter λ. The standard shift operator is denoted by T_k, i.e. $T_1\Psi(m, n, \lambda) := \Psi(m + 1, n, \lambda)$, $T_2\Psi(m, n, \lambda) := \Psi(m, n + 1, \lambda)$. To obtain $su(2)$-valued F using the Sym formula we have to use unimodular matrices:

$$\Phi := \frac{\Psi}{\sqrt{\det\Psi}} \qquad \hat{U}_k := \frac{U_k}{\sqrt{a_k^2 + (b_k\lambda + b_k'/\lambda)^2 + (c_k\lambda + c_k'/\lambda)^2 + d_k^2}}. \tag{19}$$

In this way we cancel the trace of F without changing its other components, compare [24].

Conjecture 1. *The Sym–Tafel formula $F = \Phi^{-1}\Phi_{,\lambda}$ applied to the discrete linear problem $T_k\Phi = \hat{U}_k\Phi$ yields discrete pseudospherical surfaces in arbitrary coordinates.*

An analogical conjecture has been earlier formulated in the case of local isometric immersions of Lobachevsky n-spaces in E^{2n-1} [26]. The subcase $n = 2$ corresponds to pseudospherical surfaces in curvature coordinates. However, a complete geometric description of such discretization is still missing.

Recently, Schief presented new results concerning the discretization of a large class of surfaces, including pseudospherical surfaces in curvature coordinates [27]. It would be interesting to check whether our approach, based on the Sym formula, yields the same results in this case.

4. Invariants of the Darboux–Bäcklund transformation

The Darboux–Bäcklund transformation $\tilde{\Psi} = D\Psi$ preserves, by definition, the structure of the spectral problem (e.g., its dependence on λ and group reductions). We confine ourselves to Darboux matrices of the form

$$D = \mathcal{N}\left(I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1} P\right) \tag{20}$$

where I is the unit matrix, $\mathcal{N}$ is the normalization matrix, P is a projector ($P^2 = P$) and λ_1, μ_1 are complex parameters. The projector P is expressed in terms of the background wavefunction in a well known way: $\ker P = \Psi(\lambda_1)V_{\text{ker}}$, $\operatorname{im} P = \Psi(\mu_1)V_{\text{im}}$, where V_{ker} and V_{im} are constant vector spaces, see [21]. The $su(2)$ reduction implies

$$P^\dagger = P \qquad \mathcal{N}^\dagger = \mathcal{N}^{-1} \qquad \mu_1 = \bar{\lambda}_1. \tag{21}$$

The involution (3) implies

$$P\mathbf{e}_1 = \mathbf{e}_1(I - P) \qquad \mathcal{N}\mathbf{e}_1 = \pm\mathbf{e}_1\mathcal{N} \qquad \mu_1 = -\lambda_1. \tag{22}$$

We introduce the following notation:

$$\mathrm{i}\kappa_1 := \lambda_1 = -\mu_1 \qquad Q := I - 2P. \tag{23}$$

Note that κ_1 is real and $Q^2 = I$. The reduction to the twisted $su(2)$ loop algebra implies that $Q = \sigma_2 \cos\theta + \sigma_3 \sin\theta$, where θ is a real function which, obviously, can be expressed by $\Psi(\mathrm{i}\kappa_1)$.

Taking into account (16) we rewrite the linear problem for pseudospherical surfaces in arbitrary coordinates (9) as follows:

$$\Psi_{,k} = \left(a_k\mathbf{e}_1 + \lambda\nu_k(b\mathbf{e}_2 + c\mathbf{e}_3) + \frac{\nu_k'}{\lambda}(b'\mathbf{e}_2 + c'\mathbf{e}_3)\right)\Psi. \tag{24}$$

To make the definition of ν_k, ν_k' unique we can assume (without loss of generality)

$$b^2 + c^2 = 1 \qquad (b')^2 + (c')^2 = 1 \qquad \nu_k \geqslant 0 \qquad \nu_k' \geqslant 0. \tag{25}$$

The form (24) of the linear problem is preserved by the Darboux–Bäcklund transformation (see, for instance, [21]). The explicit formulae for this transformation read

$$\begin{aligned}
&\tilde{\nu}_k(\tilde{b}\mathbf{e}_2 + \tilde{c}\mathbf{e}_3) = \nu_k\mathcal{N}(b\mathbf{e}_2 + c\mathbf{e}_3)\mathcal{N}^{-1} \\
&\tilde{\nu}_k'(\tilde{b}'\mathbf{e}_2 + \tilde{c}'\mathbf{e}_3) = \nu_k'\mathcal{N}Q(b'\mathbf{e}_2 + c'\mathbf{e}_3)Q^{-1}\mathcal{N}^{-1} \\
&\tilde{a}_k\mathbf{e}_1 = \mathcal{N}(a_k\mathbf{e}_1 - \mathrm{i}\kappa_1\nu_k[Q, b\mathbf{e}_2 + c\mathbf{e}_3])\mathcal{N}^{-1} + \mathcal{N}_{,k}\mathcal{N}^{-1}.
\end{aligned} \tag{26}$$

In the above formulae similarity transformations can be easily recognized as rotations around the $\mathbf{e}_1$-axis. Therefore the lengths of the transformed vectors are invariant, i.e.

$$\tilde{\nu}_k = \nu_k \qquad \tilde{\nu}_k' = \nu_k' \qquad (k = 1, 2). \tag{27}$$

In other words, we can consider ν_k, ν_k' as prescribed functions (more details concerning invariants of the Darboux transformation can be found in [21]).

In order to obtain a geometric interpretation of these invariants, we will take into account the compatibility conditions (15) to parametrize more explicitly the linear problem (24). Because of (25) we can introduce β, β' such that

$$b = \sin\beta \qquad c = \cos\beta \qquad b' = -\sin\beta' \qquad c' = -\cos\beta'. \tag{28}$$

Now the compatibility conditions (15) read

$$\begin{aligned} &\nu_{1,y} = \nu_{2,x} \qquad \nu'_{1,y} = \nu'_{2,x} \\ &\nu_1(a_2 - \beta_{,y}) = \nu_2(a_1 - \beta_{,x}) \\ &\nu'_1(a_2 - \beta'_{,y}) = \nu'_2(a_1 - \beta'_{,x}) \\ &a_{1,y} - a_{2,x} = (\nu_1\nu'_2 - \nu'_1\nu_2)\sin(\beta' - \beta). \end{aligned} \tag{29}$$

The first two equations imply that there exist functions $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ such that

$$\mathrm{d}\xi = \nu_1\,\mathrm{d}x + \nu_2\,\mathrm{d}y \qquad \mathrm{d}\eta = \nu'_1\,\mathrm{d}x + \nu'_2\,\mathrm{d}y. \tag{30}$$

Then the fundamental forms (12) and (13) can be written as

$$I = \mathrm{d}\xi^2 + \frac{2}{\lambda^2}\cos\phi\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{1}{\lambda^4}\,\mathrm{d}\eta^2 \qquad II = \frac{2}{\lambda}\sin\phi\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{31}$$

where $\phi := \beta' - \beta$. It means that ξ, η are asymptotic coordinates (compare (7)). Therefore the Darboux–Bäcklund transformation preserves any coordinates which are defined as given functions of asymptotic coordinates.

In particular, we obtain an obvious corollary that the reduction to asymptotic coordinates is also integrable. The condition for x, y to be asymptotic is very simple: either $\nu_1 = \nu'_2 = 0$ or $\nu_2 = \nu'_1 = 0$. Actually this case is even more evident because its defining condition consists in a different (more precise) specification of the poles of matrices of the spectral problem. Indeed, in the general situation both matrices have two poles while in the asymptotic coordinates case the first matrix has a pole at $\lambda = \infty$ and the second one has a pole at $\lambda = 0$.

The curvature coordinates diagonalize both fundamental forms. The fundamental forms (12), (13) for pseudospherical surfaces are diagonal if

$$\begin{aligned} &(b_1\lambda^2 - b'_1)(b_2\lambda^2 - b'_2) + (c_1\lambda^2 - c'_1)(c_2\lambda^2 - c'_2) = 0 \\ &b_1c'_2 + b_2c'_1 = b'_1c_2 + b'_2c_1. \end{aligned} \tag{32}$$

Let us take into account (16), (28) and assume that $F = \Psi^{-1}\Psi_{,\lambda}$ is not degenerate (i.e. $\det(I) \neq 0$, see (17)). Then equations (32) yield

$$\lambda^4\nu_1\nu_2 + \nu'_1\nu'_2 = 0 \qquad \nu'_2\nu_1 + \nu'_1\nu_2 = 0. \tag{33}$$

These constraints depend on the invariants ν_k, ν'_k and on the spectral parameter λ which is not changed by the Darboux–Bäcklund transformation as well. In the non-degenerate case ($\det(I) \neq 0$) equations (33) yield

$$\nu'_1 = \pm\lambda^2\nu_1 \qquad \nu'_2 = \mp\lambda^2\nu_2. \tag{34}$$

Corollary 1. *The Darboux–Bäcklund transformation preserves both asymptotic and curvature coordinates on pseudospherical surfaces.*

Remark 4. Note that the functions ν_k, ν'_k defining curvature coordinates depend on λ which labels a surface from Sym's λ-family. It does not contradict our earlier statement that ν_k, ν'_k do not depend on λ. In fact they depend on λ_0 (compare remark 3).

5. Bianchi surfaces, asymptotic coordinates

The strength of the compatibility condition is even more impressive in the 'non-isospectral' case. Namely, consider the linear problem (2), the same as in section 2 but now λ is not constant. We assume that λ depends not only on x and y but also on some parameter ζ. In fact ζ is the proper (constant!) spectral parameter. In the formula (1) we will replace λ by ζ.

Proposition 3. *Let the assumptions of proposition 1 be satisfied but λ is allowed to be a function $\lambda = \lambda(x, y, \zeta)$. Then $F := \Psi^{-1}\Psi,_\zeta$ has to be a Bianchi surface in $\mathbf{E}^3$ (i.e. the Gaussian curvature of F is given by $K = -1/\rho^2$ where $\rho,_{xy} = 0$) and x, y are asymptotic coordinates.*

Proof. By full analogy to the proof of proposition 1 we have

$$F,_x = \lambda,_\zeta \Psi^{-1}(B\mathbf{e}_2 + C\mathbf{e}_3)\Psi \qquad F,_y = -\lambda^{-2}\lambda,_\zeta \Psi^{-1}(Q\mathbf{e}_2 + R\mathbf{e}_3)\Psi$$

then $N = \Psi^{-1}\mathbf{e}_1\Psi$ and

$$N,_x = \Psi^{-1}(\lambda B\mathbf{e}_3 - \lambda C\mathbf{e}_2)\Psi \qquad N,_y = \Psi^{-1}(\lambda^{-1}Q\mathbf{e}_3 - \lambda^{-1}R\mathbf{e}_2)\Psi.$$

The fundamental forms can be easily computed

$$I = (\lambda,_\zeta)^2\left((B^2 + C^2)\,\mathrm{d}x^2 - \frac{2}{\lambda^2}(BQ + CR)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\lambda^4}(R^2 + Q^2)\,\mathrm{d}y^2\right)$$

$$II = \frac{2\lambda,_\zeta}{\lambda}(BR - CQ)\,\mathrm{d}x\,\mathrm{d}y.$$

The second fundamental form is off-diagonal which means that coordinates x, y are asymptotic. The Gaussian curvature of F is easily found to be

$$K = \frac{\det(I)}{\det(II)} = -\frac{\lambda^2}{(\lambda,_\zeta)^2}. \tag{35}$$

Note that to obtain this result no information about the function $\lambda = \lambda(x, y, \zeta)$ was needed. However, the compatibility conditions impose severe restrictions on the form of the function λ. Taking into account that now $\lambda \neq$ const we rewrite the compatibility conditions (4) as

$$\begin{aligned} &A,_y - P,_x + BR - CQ = 0 \\ &\lambda,_y B + \lambda(B,_y + CP) = \left(\frac{1}{\lambda}\right),_x Q + \frac{1}{\lambda}(Q,_x + AR) \\ &\lambda,_y C + \lambda(C,_y - BP) = \left(\frac{1}{\lambda}\right),_x R + \frac{1}{\lambda}(R,_x - AQ) \end{aligned} \tag{36}$$

which can be rewritten in the matrix form

$$\begin{pmatrix} B & -Q \\ C & -R \end{pmatrix}\begin{pmatrix} \lambda,_y \\ (\lambda^{-1}),_x \end{pmatrix} = \begin{pmatrix} -B,_y - CP & Q,_x + AR \\ -C,_y + BP & R,_x - AQ \end{pmatrix}\begin{pmatrix} \lambda \\ \lambda^{-1} \end{pmatrix}. \tag{37}$$

Therefore, if $QC - BR \neq 0$, we can express $\lambda,_y$ and $(\lambda^{-1}),_x$ as linear combinations of λ and λ^{-1}

$$\lambda,_y = a\lambda + b\frac{1}{\lambda} \qquad (\lambda^{-1}),_x = c\lambda + d\lambda^{-1} \tag{38}$$

where a, b, c, d do not depend on ζ. We will treat them as given functions subject to restrictions following from the compatibility conditions for (38)

$$\lambda^3(c,_y + 2ac) + \lambda(a,_x + d,_y + 4bc) + \lambda^{-1}(b,_x + 2bd) = 0$$

which split into three equations

$$b,_x + 2bd = 0 \qquad c,_y + 2ac = 0 \qquad a,_x + d,_y + 4bc = 0. \tag{39}$$

Let us try to solve the compatibility conditions (39).

If $bc \neq 0$, then

$$d = -\tfrac{1}{2}(\log|b|)_{,x} \qquad a = -\tfrac{1}{2}(\log|c|)_{,y} \tag{40}$$

$$(\log|bc|)_{,xy} = 8bc. \tag{41}$$

The last equation is equivalent to the celebrated Liouville equation $\psi_{,xy} = e^{\psi}$. Its general solution reads

$$bc = \frac{h_{,x}\, g_{,y}}{4(h+g)^2} \tag{42}$$

where $h = h(x)$ and $g = g(y)$ are arbitrary functions of one variable (the condition $bc \neq 0$ implies $h_{,x} \neq 0$ and $g_{,y} \neq 0$). Therefore,

$$b = \frac{g_{,y}}{2(g+h)\sigma} \qquad c = \frac{h_{,x}\,\sigma}{2(g+h)} \tag{43}$$

where σ is an arbitrary function of x, y. Thus the general solution of the system (39) under assumption $bc \neq 0$ is given by (40) and (43).

Taking into account this result we are going to compute λ from equations (38). Substituting (40) into (38) we obtain

$$\lambda_{,y} + \frac{c_{,y}}{2c}\lambda = \frac{b}{\lambda} \qquad \left(\frac{1}{\lambda}\right)_{,x} + \frac{b_{,x}}{2b}\frac{1}{\lambda} = c\lambda$$

and multiplying the first equation by $2c\lambda$ and the second one by $2b\lambda^{-1}$ we obtain

$$(c\lambda^2)_{,y} = 2bc \qquad (b\lambda^{-2})_{,x} = 2bc. \tag{44}$$

Rewriting (42) in the form

$$bc = -\tfrac{1}{4}(\log(h+g))_{,xy} \tag{45}$$

we can integrate once equations (44). Then, substituting (43) we get

$$h_{,x}(1+\sigma\lambda^2) = (h+g)H \qquad g_{,y}(1+(\lambda^2\sigma)^{-1}) = (h+g)G \tag{46}$$

where $H = H(x)$ and $G = G(y)$ are functions of one variable. Eliminating $\sigma\lambda^2$ we get

$$\frac{g_{,y}}{G} - g = h - \frac{h_{,x}}{H} \tag{47}$$

which implies that both sides must be constant. It is convenient to denote this constant by $1/2\zeta$. Then

$$G = \frac{2\zeta g_{,y}}{2\zeta g + 1} \qquad H = \frac{2\zeta h_{,x}}{2\zeta h - 1} \tag{48}$$

where $\zeta = \text{const}$. Finally, from (46) we compute

$$\sigma\lambda^2 = \frac{2\zeta g + 1}{2\zeta h - 1}. \tag{49}$$

Remark 5. The linear system (2) is invariant under the transformation $\lambda \to f(x, y)\lambda$ (for any given function f) and also under the transformation $\lambda \to 1/\lambda$, $x \leftrightarrow y$. Therefore σ can be prescribed arbitrarily without loss of generality (the second transformation can be necessary because the first transformation cannot change the sign of σ).

Note that

$$\hat{\rho} := \frac{\lambda_{,\zeta}}{\lambda} = \frac{g}{1+2\zeta g} + \frac{h}{1-2\zeta h} \tag{50}$$

which means (compare (35)) that the Gaussian curvature $K \equiv -\hat{\rho}^{-2}$ does not depend on σ and, moreover, $\hat{\rho}_{,xy} = 0$. Equations (38) assume the form

$$\begin{aligned} \lambda_{,x} &= -\frac{h_{,x}}{2(h+g)}(\sigma\lambda^3 + \lambda) - \frac{\lambda\sigma_{,x}}{2\sigma} \\ \lambda_{,y} &= \frac{g_{,y}}{2(h+g)}\left(\lambda + \frac{1}{\sigma\lambda}\right) - \frac{\lambda\sigma_{,y}}{2\sigma}. \end{aligned} \tag{51}$$

Now, we take into account remark 5. Fixing $\sigma = -1$ and denoting

$$\rho(x, y) := h(x) + g(y) \tag{52}$$

we represent these equations in a more familiar way (compare [18, 19]):

$$\lambda_{,x} = \frac{\rho_{,x}}{2\rho}(\lambda^3 - \lambda) \qquad \lambda_{,y} = \frac{\rho_{,y}}{2\rho}(\lambda - \lambda^{-1}) \tag{53}$$

with general solution given by

$$\lambda^2 = \frac{1 + 2\zeta g}{1 - 2\zeta h}. \tag{54}$$

In the case $g \equiv 0$ another choice can be more natural, namely $\sigma = 1/h$. Then

$$\lambda_{,x} = -\frac{h_{,x}\lambda^3}{2h^2} \qquad \lambda_{,y} = 0. \tag{55}$$

Similarly, in the case $h \equiv 0$ we can take $\sigma = g$ and

$$\lambda_{,y} = \frac{g_{,y}}{2g^2\lambda} \qquad \lambda_{,x} = 0. \tag{56}$$

Such dependence of the spectral parameter has been assumed, for instance, by Schief [17, 28].

In the case $b = 0$, $c \neq 0$ equations (39) and (38) yield

$$\begin{aligned} a &= -\frac{1}{2}(\log c)_{,y} & d &= -\frac{1}{2}(\log c)_{,x} + \phi \\ \lambda_{,y} &= -\frac{1}{2}(\log c)_{,y}\lambda & \lambda_{,x} &= -c\lambda^3 - \left(\frac{c_{,x}}{2c} + \phi\right)\lambda \end{aligned}$$

where $\phi = \phi(x)$ is a function of one variable. There is no constraint on c, i.e. $c = c(x, y)$ is an arbitrary function. The equations for λ can be represented in the form

$$(\lambda^2 c)_{,y} = 0 \qquad (\lambda^2 c)_{,x} + 2(\lambda^2 c)^2 + 2\phi\lambda^2 c = 0 \tag{57}$$

and can be explicitly solved. Indeed, introducing functions χ, ψ

$$\chi := \frac{1}{\lambda^2 c} \qquad 2\phi = -(\log\psi)_{,x}$$

we transform (57) into

$$\chi_{,y} = 0 \qquad (\psi\chi)_{,x} = 2\psi. \tag{58}$$

Integrating, we solve the equation for χ to get

$$\frac{1}{c\lambda^2} = \frac{\zeta}{\psi} + \frac{2}{\psi}\int_{x_0}^{x} \psi(\xi)\,\mathrm{d}\xi$$

where ζ is the constant of the integration. To obtain a formula similar to (49) we denote

$$\sigma(x, y) := -\frac{2c(x, y)}{\psi(x)}\int_{x_0}^{x} \psi(\xi)\,\mathrm{d}\xi \qquad h(x) := -\left(4\int_{x_0}^{x} \psi(\xi)\,\mathrm{d}\xi\right)^{-1}.$$

Because ψ and c are arbitrary functions, σ and h are arbitrary as well. Therefore

$$\sigma\lambda^2 = \frac{1}{2\zeta h - 1} \tag{59}$$

which is a special case of (49) ($g = 0$) as should be expected. Equation (50), with $g = 0$, is satisfied as well. The Gaussian curvature, given by

$$K = -(h^{-1} - 2\zeta)^2$$

is a function of one variable (compare [28]).

In the last case (i.e. $b = c = 0$) we have

$$a = \phi,_y \qquad d = -\phi,_x \qquad \lambda,_y = \phi,_y \lambda \qquad \lambda,_x = \phi,_x \lambda$$

which means that $\lambda = \zeta\, e^{\phi}$ (where $\zeta =$ const) and, finally, $K = -\zeta^2 =$ const. Thus the proof is completed. □

Remark 6. It is well known that the form (2) of the spectral problem and the dependence of λ on x, y given by (53) correspond to Bianchi surfaces equipped with asymptotic coordinates (compare [19]). A novelty consists in deriving the formula (49) directly from the compatibility conditions as the unique possibility.

The discretization of Bianchi surfaces in asymptotic coordinates has been proposed recently [29]. A derivation of this result based on geometrical considerations will be published soon [30]. The corresponding discrete system was first obtained (and shown to be integrable) in the context of isothermic surfaces [7]. A discretization of Bianchi surfaces in any other coordinates is still an open problem.

6. Bianchi surfaces, arbitrary coordinates

Performing an arbitrary change of coordinates for the system (2) we obtain linear equations which are both linear in λ and λ^{-1}. It is surprising however that any linear problem of this kind is associated with Bianchi surfaces. Indeed, let us consider an $su(2)$ spectral problem

$$\Psi,_k = \left(\mathbf{A}_k + \lambda \mathbf{B}_k + \frac{1}{\lambda}\mathbf{C}_k\right)\Psi \qquad (k = 1, 2) \tag{60}$$

where we assume the reduction $U_k(-\lambda)\mathbf{e}_1 = \mathbf{e}_1 U_k(\lambda)$. In other words,

$$\mathbf{A}_k := a_k \mathbf{e}_1 \tag{61}$$

and $\mathbf{B}_k$, $\mathbf{C}_k$ are λ-independent linear combinations of $\mathbf{e}_2$ and $\mathbf{e}_3$.

The compatibility conditions read

$$\begin{aligned} &\mathbf{A}_1,_2 + (\lambda\mathbf{B}_1),_2 + (\lambda^{-1}\mathbf{C}_1),_2 - \mathbf{A}_2,_1 - (\lambda\mathbf{B}_2),_1 - (\lambda^{-1}\mathbf{C}_2),_1 \\ &\quad + [\mathbf{A}_1, \mathbf{A}_2] + \lambda^2[\mathbf{B}_1, \mathbf{B}_2] + \lambda^{-2}[\mathbf{C}_1, \mathbf{C}_2] + [\mathbf{B}_1, \mathbf{C}_2] + [\mathbf{C}_1, \mathbf{B}_2] \\ &\quad + \lambda([\mathbf{A}_1, \mathbf{B}_2] + [\mathbf{B}_1, \mathbf{A}_2]) + \lambda^{-1}([\mathbf{A}_1, \mathbf{C}_2] + [\mathbf{C}_1, \mathbf{A}_2]) = 0. \end{aligned} \tag{62}$$

The coefficients by $\lambda^2\mathbf{e}_1$ and $\lambda^{-2}\mathbf{e}_1$ imply $[\mathbf{B}_1, \mathbf{B}_2] = [\mathbf{C}_1, \mathbf{C}_2] = 0$ which means that

$$\mathbf{B}_k = B_k\mathbf{B} \qquad \mathbf{C}_k = C_k\mathbf{C} \qquad (k = 1, 2) \tag{63}$$

where B_k and C_k are some scalar functions and $\mathbf{B}$ and $\mathbf{C}$ are of unit length, i.e.

$$\mathbf{B} = \mathbf{e}_2\cos\beta + \mathbf{e}_3\sin\beta \qquad \mathbf{C} = \mathbf{e}_2\cos\gamma + \mathbf{e}_3\sin\gamma. \tag{64}$$

Thus

$$\langle\mathbf{B}|\mathbf{C}\rangle = \cos\varphi \qquad [\mathbf{B}, \mathbf{C}] = \mathbf{e}_1\sin\varphi \tag{65}$$

where $\varphi := \gamma - \beta$. Then the $\mathbf{e}_1$-component of (62) yields

$$a_{1,2} - a_{2,1} + (B_1 C_2 - C_1 B_2) \sin\varphi = 0. \tag{66}$$

Consider the part of the compatibility conditions (62) given by coefficients $\mathbf{e}_2$ and $\mathbf{e}_3$. If $\mathbf{B}$ and $\mathbf{C}$ are linearly independent, then it is convenient to use them as a basis. Let us decompose the following expressions with respect to this basis:

$$(B_1\mathbf{B})_{,2} - (B_2\mathbf{B})_{,1} + (a_1 B_2 - a_2 B_1)[\mathbf{e}_1, \mathbf{B}] = p_1\mathbf{B} + p_2\mathbf{C}$$
$$(C_1\mathbf{C})_{,2} - (C_2\mathbf{C})_{,1} + (a_1 C_2 - a_2 C_1)[\mathbf{e}_1, \mathbf{C}] = q_1\mathbf{C} + q_2\mathbf{B}$$

where $p_k = p_k(x^1, x^2)$ and $q_k = q_k(x^1, x^2)$ are the corresponding components which can be easily computed:

$$p_2 \sin\varphi = B_1\beta_{,2} - B_2\beta_{,1} + a_1 B_2 - a_2 B_1$$
$$q_2 \sin\varphi = C_2\gamma_{,1} - C_1\gamma_{,2} + a_2 C_1 - a_1 C_2$$
$$p_1 = B_{1,2} - B_{2,1} - p_2\cos\varphi$$
$$q_1 = C_{1,2} - C_{2,1} - q_2\cos\varphi.$$

Then this part of the compatibility conditions reads

$$\left(\lambda_{,2} B_1 - \lambda_{,1} B_2 + \lambda p_1 + \frac{1}{\lambda} p_2\right)\mathbf{B} + \left(\frac{\lambda_{,1}}{\lambda^2} C_2 - \frac{\lambda_{,2}}{\lambda^2} C_1 + \lambda q_2 + \frac{1}{\lambda} q_1\right)\mathbf{C} = 0.$$

Solving them with respect to derivatives of λ we get

$$\begin{pmatrix}\lambda_{,1}\\ \lambda_{,2}\end{pmatrix} = \frac{1}{B_2 C_1 - B_1 C_2}\begin{pmatrix} B_1 q_2 \lambda^3 + (C_1 p_1 + B_1 q_1)\lambda + C_1 p_2 \lambda^{-1} \\ B_2 q_2 \lambda^3 + (C_2 p_1 + B_2 q_1)\lambda + C_2 p_2 \lambda^{-1}\end{pmatrix}. \tag{67}$$

Introducing the notation

$$\zeta = \lambda^2$$

we immediately see that (67) is a system of two Riccati equations for ζ:

$$\zeta_{,k} = 2\alpha_k \zeta^2 + 2\beta_k \zeta + 2\gamma_k \qquad (k = 1, 2) \tag{68}$$

where

$$\alpha_k = \frac{B_k q_2}{B_2 C_1 - B_1 C_2} \qquad \beta_k = \frac{(C_k p_1 + B_k q_1)}{B_2 C_1 - B_1 C_2} \qquad \gamma_k = \frac{C_k p_2}{B_2 C_1 - B_1 C_2}. \tag{69}$$

The Riccati equations for ζ are more general than the previous system (53). However, they can be reduced to (53) by a change of variables. It is enough to show that after a change of variables γ_1 and α_2 vanish.

Proposition 4. *The change of variables $(x^1, x^2) \mapsto (\tilde{x}^1, \tilde{x}^2)$ defined by*

$$C_1 \frac{\partial x^1}{\partial \tilde{x}^1} + C_2 \frac{\partial x^2}{\partial \tilde{x}^1} = 0 \qquad B_1 \frac{\partial x^1}{\partial \tilde{x}^2} + B_2 \frac{\partial x^2}{\partial \tilde{x}^2} = 0 \tag{70}$$

transforms the spectral problem (60) into (2), i.e. $\tilde{\mathbf{C}}_1 = \tilde{\mathbf{B}}_2 = 0$. Moreover, $\tilde{\gamma}_1 = \tilde{\alpha}_2 = 0$.

The proof is straightforward. Let us point out that the condition for vanishing $\tilde{\gamma}_1$ and $\tilde{\alpha}_2$ is of the form

$$\gamma_1 \frac{\partial x^1}{\partial \tilde{x}^1} + \gamma_2 \frac{\partial x^2}{\partial \tilde{x}^1} = 0 \qquad \alpha_1 \frac{\partial x^1}{\partial \tilde{x}^2} + \alpha_2 \frac{\partial x^2}{\partial \tilde{x}^2} = 0.$$

These equations follow from (70) because of (69).

Let us proceed to describe the geometry of the corresponding Sym's surfaces. Defining, as in the previous section, $F := \Psi^{-1}\Psi_{,\zeta}$ we have

$$F_{,k} = \lambda_{,\zeta}\Psi^{-1}(B_k\mathbf{B} - \lambda^{-2}C_k\mathbf{C})\Psi \tag{71}$$

and the coefficients of the first fundamental form read

$$\begin{aligned} g_{11} &= \langle F_{,1}|F_{,1}\rangle = (\lambda_{,\zeta})^2\left(B_1^2 - 2\lambda^{-2}B_1C_1\cos\varphi + \lambda^{-4}C_1^2\right) \\ g_{12} &= \langle F_{,1}|F_{,2}\rangle = (\lambda_{,\zeta})^2(B_1B_2 - \lambda^{-2}(B_1C_2 + B_2C_1)\cos\varphi + \lambda^{-4}C_1C_2) \\ g_{22} &= \langle F_{,2}|F_{,2}\rangle = (\lambda_{,\zeta})^2\left(B_2^2 - 2\lambda^{-2}B_2C_2\cos\varphi + \lambda^{-4}C_2^2\right). \end{aligned} \tag{72}$$

The determinant $\det(I) \equiv g_{11}g_{22} - g_{12}^2$ is given by

$$\det(I) = \left(\frac{\lambda_{,\zeta}}{\lambda}\right)^4 (C_1B_2 - C_2B_1)^2\sin^2\varphi. \tag{73}$$

Computing

$$[F_{,1}, F_{,2}] = \left(\frac{\lambda_{,\zeta}}{\lambda}\right)^2 (C_1B_2 - B_1C_2)\Psi^{-1}[\mathbf{B}, \mathbf{C}]\Psi$$

we see (compare (65)) that the normal vector is given by

$$N = \Psi^{-1}\mathbf{e}_1\Psi. \tag{74}$$

To obtain the coefficients of the second fundamental form, $\mathrm{B}_{jk} = -\langle F_{,j}|N_{,k}\rangle$, we compute

$$N_{,k} = \Psi^{-1}[\mathbf{e}_1, \mathbf{A}_k + \lambda\mathbf{B}_k + \lambda^{-1}\mathbf{C}_k]\Psi = \Psi^{-1}(\lambda B_k[\mathbf{e}_1, \mathbf{B}] + \lambda^{-1}C_k[\mathbf{e}_1, \mathbf{C}])\Psi. \tag{75}$$

Then, taking into account $\langle\mathbf{B}\,|[\mathbf{e}_1, \mathbf{B}]\rangle = \langle\mathbf{C}\,|[\mathbf{e}_1, \mathbf{C}]\rangle = 0$, we have

$$\langle F_{,j}|N_{,k}\rangle = (\lambda_{,\zeta}/\lambda)(B_jC_k\langle\mathbf{B}|[\mathbf{e}_1, \mathbf{C}]\rangle - C_jB_k\langle\mathbf{C}|[\mathbf{e}_1, \mathbf{B}]\rangle.$$

Moreover, $\langle\mathbf{C}|[\mathbf{e}_1, \mathbf{B}]\rangle = -\langle\mathbf{B}\,|[\mathbf{e}_1, \mathbf{C}]\rangle = \sin\varphi$. Therefore

$$II = 2(\lambda_{,\zeta}/\lambda)\sin\varphi(B_1C_1\,\mathrm{d}x^2 + (B_1C_2 + B_2C_1)\,\mathrm{d}x\,\mathrm{d}y + B_2C_2\,\mathrm{d}y^2) \tag{76}$$

and $\det(II) = -(\lambda_{,\zeta}/\lambda)^2\sin^2\varphi(B_1C_2 - B_2C_1)^2$. Finally,

$$K = \frac{\det(I)}{\det(II)} = -\left(\frac{\lambda_{,\zeta}}{\lambda}\right)^2. \tag{77}$$

Therefore, the case studied in this section corresponds exactly to Bianchi surfaces endowed with arbitrary coordinates.

The Darboux–Bäcklund transformation given by (20) for the linear problem (60) (with the λ-dependence given by (68)) reads (compare [21])

$$\begin{aligned} \mathcal{N}^{-1}\tilde{\mathbf{A}}_k\mathcal{N} &= \mathbf{A}_k - \mathrm{i}\kappa_1[Q, \mathbf{B}_k] + \mathcal{N}^{-1}\mathcal{N}_{,k} \\ \mathcal{N}^{-1}\tilde{\mathbf{B}}_k\mathcal{N} &= \mathbf{B}_k + \mathrm{i}\kappa_1\alpha_kQ \\ \mathcal{N}^{-1}\tilde{\mathbf{C}}_k\mathcal{N} &= Q\mathbf{C}_kQ + \mathrm{i}\kappa_1^{-1}\gamma_kQ \end{aligned} \tag{78}$$

where $\mathcal{N}, Q, \kappa_1$ are defined in section 4. In the non-isospectral case the $\mathrm{i}\kappa_1$ is a function satisfying the same differential equation as λ (see [21]).

The Darboux–Bäcklund transformation in this case does not seem to have as many invariants as in the isospectral case (compare section 4). We just remark that the constraints

$$\mathbf{C}_1 = \mathbf{B}_2 = 0 \qquad \gamma_1 = \alpha_2 = 0$$

are obviously invariant with respect to the transformation (78), i.e. $\tilde{\mathbf{C}}_1 = \tilde{\mathbf{B}}_2 = 0$. These constraints reduce the spectral problem (60) to the non-isospectral version of (2) which corresponds to pseudospherical surfaces in asymptotic coordinates. Asymptotic coordinates are also obtained for the analogical ('symmetric') case: $\mathbf{C}_2 = \mathbf{B}_1 = 0$, $\gamma_2 = \alpha_1 = 0$.

7. Physical applications

In the framework of the soliton surfaces approach the spectral problems considered in this paper can be associated with some models of physical interest.

The position vector $F = F(x, y; \zeta)$ to Bianchi surfaces in asymptotic coordinates (compare proposition 3) satisfies the equation

$$F_{,xy} = -\frac{1}{\hat{\rho}} F_{,x} \times F_{,y} + \frac{\hat{\rho}_{,y}}{2\hat{\rho}} F_{,x} + \frac{\hat{\rho}_{,x}}{2\hat{\rho}} F_{,y} \qquad \hat{\rho}_{,xy} = 0 \tag{79}$$

where $\hat{\rho} = \lambda_{,\zeta}/\lambda$ (see also (50)) and the cross means the vector (skew) product in $\mathbf{E}^3$. Note that for $\zeta = 0$ we have $\hat{\rho} = \rho = h(x) + g(y)$.

The physical interpretation of equation (79) is clear especially in the case $\hat{\rho} = \text{const}$ [18] when it represents a Lund–Regge vortex model [31] or relativistic string moving in the constant external field [32]. This is a chiral model, its solutions are harmonic maps from two-dimensional Minkowski space into a two-dimensional submanifold of E^3. The additional terms appearing for $\hat{\rho} \neq \text{const}$ can be associated with a friction proportional to the velocity [18].

In terms of the tangent vectors, $S := F_{,x}$ and $T := F_{,y}$, we can rewrite (79) as

$$S_{,y} = \frac{1}{\hat{\rho}} T \times S + \frac{\hat{\rho}_{,y}}{2\hat{\rho}} S + \frac{\hat{\rho}_{,x}}{2\hat{\rho}} T \qquad T_{,x} = S_{,y} \qquad \hat{\rho}_{,xy} = 0 \tag{80}$$

(note that the above form is simpler than, although equivalent to, that given in [18] and correct, in contrast to the misprinted formula given in [21]). The specialization $\hat{\rho} = \text{const}$ reduces this system to a quantum optics model [33].

However, the Bianchi surfaces yield but a subset of solutions to equation (79). Indeed, the $su(2)$ spectral problem of the general form

$$\Psi_{,x} = (U_0 + \lambda U_1)\Psi \qquad \Psi_{,y} = \left(V_0 + \frac{1}{\lambda} V_1\right)\Psi \tag{81}$$

with λ depending on x and y generates solutions to equation (79) as well. This result has not been fully discussed in the literature yet (some results for $\hat{\rho} = \text{const}$ can be found in [32]), so we give here more details. The compatibility conditions, similarly as in section 5, imply equations (53) and (54) for x, y-dependence of λ (ρ is a given function satisfying $\rho_{,xy} = 0$) and the following nonlinear system for U_0, V_0, U_1, V_1:

$$\begin{aligned} &U_{1,y} + [U_1, V_0] + \frac{\rho_{,y}}{2\rho} U_1 + \frac{\rho_{,x}}{2\rho} V_1 = 0 \\ &V_{1,x} + [U_0, V_1] + \frac{\rho_{,y}}{2\rho} U_1 + \frac{\rho_{,x}}{2\rho} V_1 = 0 \\ &U_{0,y} - V_{0,x} + [U_0, V_0] + [U_1, V_1] = 0. \end{aligned} \tag{82}$$

Then $F = \Psi^{-1}\Psi_{,\zeta}$ yields

$$\begin{aligned} &F_{,x} = \lambda_{,\zeta}\Psi^{-1}U_1\Psi \qquad F_{,y} = -\lambda_{,\zeta}\lambda^{-2}\Psi^{-1}V_1\Psi \\ &F_{,xy} = \lambda_{,\zeta}\Psi^{-1}\left(U_{1,y} + \frac{\rho_{,y}}{2\rho}U_1\left(1 + \frac{1}{\lambda^2}\right) + [U_1, V_0] + \frac{1}{\lambda}[U_1, V_1]\right)\Psi. \end{aligned}$$

Using the compatibility conditions, taking into account that

$$\frac{\rho_{,x}\lambda^2}{2\rho} = \frac{\hat{\rho}_{,x}}{2\hat{\rho}} \qquad \frac{\rho_{,y}}{2\rho\lambda^2} = \frac{\hat{\rho}_{,y}}{2\hat{\rho}}$$

and identifying the commutator with the skew product (compare remark 2) we get (79).

Therefore the 'twisting' condition (3), very important for geometrical reasons, is not necessary for producing equation (79).

The following constraints on the system (82)

$$\hat{\rho}_{,x} = 0 \qquad V_0 = 0 \qquad U_1 = f(t)\mathbf{e}_3 \qquad U_0 \perp U_1 \qquad y \equiv t \tag{83}$$

yield $\rho f^2 =$ const and reduce equations (82) to the pumped Maxwell–Bloch system [17]:

$$E_{,t} = P \qquad P_{,x} = En \qquad n_{,x} + \tfrac{1}{2}(\bar{E}P + E\bar{P}) = ff_{,t} \tag{84}$$

where $f = f(t)$ is a given function. To obtain the system (84) we have to parametrize U_0 and V_1 as follows:

$$U_0 = \mathrm{Re}(E)\mathbf{e}_1 + \mathrm{Im}(E)\mathbf{e}_2 \qquad fV_1 = -\mathrm{Im}(P)\mathbf{e}_1 + \mathrm{Re}(P)\mathbf{e}_2 + n\mathbf{e}_3$$

where complexes E and P denote the envelope amplitude of the electric field and polarization, respectively, n is the atomic inversion and x, t are scaled space and retarded time variables. The case $f =$ const yields an unpumped Maxwell–Bloch system, equivalent to sharpline self-induced transparency equations [34, 35]. For f^2 linear in t the pumping is constant [17].

The involution (3) defining Bianchi surfaces imposes the reality conditions on the solutions of the Maxwell–Bloch system: $\mathrm{Im}(E) = 0, \mathrm{Im}(P) = 0$. Therefore real solutions of the pumped Maxwell–Bloch system 'live' on Bianchi surfaces with curvature depending on just one variable, $K = K(x)$. The real solutions of the unpumped system are associated with the pseudospherical surfaces. The same concerns pure imaginary solutions (it is enough to take another involution, replacing $\mathbf{e}_1$ by $\mathbf{e}_2$).

The constraints (83) are preserved by the Darboux–Bäcklund transformation. Indeed, the constraints for U_1 and U_0 are standard (see, for instance, [21]). The condition $V_0 = 0$ can be rewritten in this case as $V(\lambda) \to 0$ for $\lambda \to \infty$. Such a condition is always preserved provided that the normalization matrix $\mathcal{N}$ is constant.

Another physical connection arises when we consider the kinematics of the normal vector $N = (N_1, N_2, N_3)$ to the Bianchi surface (in asymptotic coordinates, $x^1 = x, x^2 = y$) given by the Sym formula $F = \Psi^{-1}\Psi_{,\zeta}$

$$N_{,12} + \frac{\hat{\rho}_{,2}}{2\hat{\rho}}N_{,1} + \frac{\hat{\rho}_{,1}}{2\hat{\rho}}N_{,2} + CN = 0 \qquad N^2 = 1 \qquad \hat{\rho}_{,12} = 0 \tag{85}$$

where the scalar function C can be easily expressed in terms of N using the constraint $N^2 = 1$. Denoting $n := N\sqrt{\hat{\rho}}$ we can rewrite (85) as the Moutard equation

$$n_{,12} = \Lambda n \qquad n^2 = \hat{\rho} \tag{86}$$

where $\hat{\rho}$ is a given function satisfying $\hat{\rho}_{,12} = 0$. The corresponding Lagrangian is given by [21]

$$\mathcal{L} = \frac{1}{2}\int (\langle n_{,1}|n_{,2}\rangle + \Lambda(\langle n|n\rangle - \hat{\rho}))\,\mathrm{d}x^1\,\mathrm{d}x^2. \tag{87}$$

Equation (85) is closely related to the Ernst equation describing a stationary, axially symmetric gravitational field in general relativity [36]:

$$\mathcal{E}_{,z\bar{z}} + \frac{\rho_{,\bar{z}}}{2\rho}\mathcal{E}_{,z} + \frac{\rho_{,z}}{2\rho}\mathcal{E}_{,\bar{z}} = \frac{\mathcal{E}_{,z}\,\mathcal{E}_{,\bar{z}}}{\mathrm{Re}(\mathcal{E})} \qquad \rho_{,z\bar{z}} = 0 \tag{88}$$

(z and $\mathcal{E}$ are complex). Indeed, using a stereographic projection, namely introducing the complex variable

$$\mathcal{E} = \frac{N_1 + \mathrm{i}N_2}{1 + N_3} \tag{89}$$

we can rewrite (85) in the form

$$\mathcal{E}_{,12} + \frac{\hat{\rho}_{,2}}{2\hat{\rho}}\mathcal{E}_{,1} + \frac{\hat{\rho}_{,1}}{2\hat{\rho}}\mathcal{E}_{,2} = \frac{2\mathcal{E}_{,1}\mathcal{E}_{,2}\bar{\mathcal{E}}}{1+|\mathcal{E}|^2} \qquad \hat{\rho}_{,12} = 0 \tag{90}$$

which is quite similar to (88). To make this analogy even closer, one can represent (85) as the equation of the following chiral model (or sigma model):

$$(\hat{\rho}\mathbf{n}^{-1}\mathbf{n}_{,1})_{,2} + (\hat{\rho}\mathbf{n}^{-1}\mathbf{n}_{,2})_{,1} = 0 \qquad \mathbf{n}^2 = -1 \qquad \hat{\rho}_{,12} = 0 \tag{91}$$

where in the case of the Bianchi surfaces $\mathbf{n}$ is just the $su(2)$ counterpart of N, i.e. $\mathbf{n} = \mathrm{i}\sigma_1 N_1 + \mathrm{i}\sigma_2 N_2 + \mathrm{i}\sigma_3 N_3$ (compare remark 2). This is a hyperbolic sigma model with the symmetry $su(2)$ (or $so(3)$).

The Ernst equation is equivalent to the same sigma model (91) with just two differences: it is elliptic (i.e. instead of real independent variables we have complex variables, $x^1 = z$, $x^2 = \bar{z}$) with the symmetry $so(2, 1)$ (for more details, see [35]).

8. Conclusions

In the theory of solitons the construction of integrable systems starting from some assumptions on the form of spectral problem is standard. We exploit this approach in the field of classical geometry of immersed surfaces using Sym's approach of soliton surfaces. The main point was to weaken the assumptions as much as possible. The results are very promising. Spectral problems based on the twisted $su(2)$ loop algebra with at most two poles lead uniquely to pseudospherical surfaces and (in the non-isospectral case) to Bianchi surfaces.

The approach of this kind is very convenient if the construction of the Darboux–Bäcklund transformation is concerned (compare [21]). Indeed, such construction is very well known and standard in the case of spectral problems in a general form. Some problems can be created by reductions (except group reductions which are also relatively well known). Minimizing the number of restrictions on the form of the spectral problem certainly helps to construct the Darboux–Bäcklund transformation.

However, the most challenging problem is to find discrete versions of the presented spectral problems. Thus we will be able to characterize discrete hyperbolic (in particular, pseudospherical) surfaces in a coordinate-independent way.

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References

[1] Bobenko A I 1994 Surfaces in terms of 2 by 2 matrices Old and new integrable cases *Harmonic Maps and Integrable Systems (Aspects of Mathematics vol 23)* ed A P Fordy and J C Wood (Brunswick: Vieweg)

[2] Fokas A S and Gelfand I M 1996 Surfaces on Lie groups, on Lie algebras and their integrability *Commun. Math. Phys.* **177** 203–20

[3] Cieśliński J 1997 A generalized formula for integrable classes of surfaces in Lie algebras *J. Math. Phys.* **38** 4255–72

[4] Wójcik D and Cieśliński J (ed) 1998 *Nonlinearity & Geometry* (Warsaw: Polish Scientific Publishers PWN)

[5] Dodd R K 1998 Soliton immersions *Commun. Math. Phys.* **197** 641–65

[6] Konopelchenko B G and Landolfi G 1999 Generalized Weierstrass representation for surfaces in multi-dimensional Riemann spaces *J. Geom. Phys.* **29** 319–33

[7] Schief W K 2001 Isothermic surfaces in spaces of arbitrary dimension: integrability, discretization, and Bäcklund transformations—a discrete Calapso equation *Stud. Appl. Math.* **106** 85–137

[8] Eisenhart L P 1909 *A Treatise on the Differential Geometry of Curves and Surfaces* (Boston, MA: Ginn) (reprinted 1960 (New York: Dover))

[9] Chern S S 1991 Surface theory with Darboux and Bianchi *Miscellanea Mathematica* ed P Hilton, F Hirzebruch and R Remmert (Berlin: Springer) pp 59–69

[10] Sym A 1982 Soliton surfaces *Lett. Nuovo Cimento* **33** 394–400

[11] Sym A 1985 Soliton surfaces and their application. Soliton geometry from spectral problems *Geometric Aspects of the Einstein Equations and Integrable Systems (Lecture Notes in Physics 239)* ed R Martini (Berlin: Springer) pp 154–231

[12] Fokas A S, Gelfand I M, Finkel F and Liu Q M 2000 A formula for constructing infinitely many surfaces on Lie algebras and integrable equations *Sel. Math. New Ser.* **6** 347–75

[13] Cieśliński J L 2000 A class of linear spectral problems in Clifford algebras *Phys. Lett.* A **267** 251–5

[14] Estabrook F B 1982 Moving frames and prolongation algebras *J. Math. Phys.* **23** 2071–6

[15] Harrison B K 1983 Unification of Ernst equation Bäcklund transformations using a modified Wahlquist–Estabrook technique *J. Math. Phys.* **24** 2178–87

[16] Hoenselaers C 1985 The sine-Gordon prolongation algebra *Progr. Theor. Phys.* **74** 645–54

[17] Schief W K 1994 Bäcklund transformation for the (un)pumped Maxwell–Bloch system and the fifth Painlevé equation *J. Phys. A: Math. Gen.* **27** 547–57

[18] Levi D and Sym A 1990 Integrable systems describing surfaces of non-constant curvature *Phys. Lett.* A **149** 381–7

[19] Schief W K, Rogers C and Johnston M E 1995 On hyperbolic surfaces: moving triad and Loewner system connections *Chaos, Solitons Fractals* **5** 25–34

[20] Tafel J 1995 Surfaces in R^3 with prescribed curvature *J. Geom. Phys.* **17** 381–90

[21] Cieśliński J 1995 An algebraic method to construct the Darboux matrix *J. Math. Phys.* **36** 5670–706

[22] Korotkin D 1996 On some integrable cases in surface theory *Zap. Nauch. Sem. POMI* **15** 65–124 (in Russian) (Engl. transl. 1999 *J. Math. Sci. (NY)* **94** 1177–217)

[23] Sym A 1983 Soliton surfaces: II. Geometric unification of solvable nonlinearities *Lett. Nuovo Cimento* **36** 307–12

[24] Bobenko A and Pinkall U 1996 Discrete surfaces with constant negative Gaussian curvature and the Hirota equation *J. Diff. Geom.* **43** 527–611

[25] Bobenko A and Pinkall U 1996 Discrete isothermic surfaces *J. Reine Angew. Math.* **475** 187–208

[26] Cieśliński J 1997 The spectral interpretation of n-spaces of constant negative curvature immersed in $\mathbf{R}^{2n-1}$ *Phys. Lett.* A **236** 425–30

[27] Schief W K 2001 On the unification of classical and novel integrable surfaces: II. Difference geometry *Preprint* nlin.SI/0104037

[28] Schief W K 1995 An infinite hierarchy of symmetries associated with hyperbolic surfaces *Nonlinearity* **8** 1–9

[29] Doliwa A, Nieszporski M and Santini P M 2001 Asymptotic lattices and their integrable reductions: I. The Bianchi-Ernst and the Fubini-Ragazzi lattices *J. Phys. A: Math. Gen.* **34** 10423–39

[30] Doliwa A, Nieszporski M and Santini P M Geometric discretization of the Bianchi system (submitted)

[31] Lund F and Regge T 1976 Unified approach to strings and vortices with soliton solutions *Phys. Rev.* D **14** 1524–35

[32] Sym A, Ragnisco O, Levi D and Bruschi M 1985 Soliton surfaces VII. Relativistic string in external field: general integral and particular solutions *Lett. Nuovo Cimento* **44** 529–36

[33] Tratnik M V and Sipe J E 1987 Polarization solitons *Phys. Rev. Lett.* **58** 1104–07

[34] Steudel H 1991 Space-time symmetry of self-induced transparency and of stimulated Raman scattering *Phys. Lett.* A **156** 491–2

[35] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory* (Cambridge: Cambridge University Press)

[36] Ernst F 1968 New formulation of the axially symmetric gravitational field problem. II *Phys. Rev.* **168** 1415–7